

Elementary Derivation of the Dirac Equation. X

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Final considerations of the isomorphism between electrodynamics and wave mechanics.

1. The Maxwell-Dirac Isomorphism

The preceding nine papers of ours were concerned with the formal connection between electrodynamics and wave mechanics, a problem which has attracted the attention of a number of other authors too [1–6]. We were able to demonstrate the isomorphism of the two theories in the domain of differential equations of the first order. There we ultimately started from the completely source-free electrodynamics, i.e.

$$\text{rot } \mathbf{E} + \frac{\mu}{c} \frac{\partial}{\partial t} \mathbf{H} = \mathbf{0}, \quad \text{div } \varepsilon \mathbf{E} = 0, \quad \text{div } \mathbf{E} = 0, \quad (1)$$

$$\text{rot } \mathbf{H} - \frac{\varepsilon}{c} \frac{\partial}{\partial t} \mathbf{E} = \mathbf{0}, \quad \text{div } \mu \mathbf{H} = 0, \quad \text{div } \mathbf{H} = 0.$$

Skalar-multiplication of these equations by the Pauli-vector leaves the last four unchanged, while the first two go over into

$$\left[\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \cdot \nabla - \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \frac{\partial}{\partial c t} \right] \begin{bmatrix} i(\boldsymbol{\sigma} \cdot \mathbf{E}) \\ (\boldsymbol{\sigma} \cdot \mathbf{H}) \end{bmatrix}. \quad (2)$$

This coincides, after separating out the time dependence, with the Dirac amplitude equation

$$\left[\left(\boldsymbol{\gamma} \cdot \nabla + \frac{i}{\hbar c} \begin{pmatrix} \hbar \omega - \Phi + m_0 c^2 & 0 \\ 0 & \hbar \omega - \Phi - m_0 c^2 \end{pmatrix} \right) \right] \Psi = 0. \quad (4)$$

The electrodynamical as well as the wave-mechanical components are interconnected, as are refraction (ε, μ) and potential Φ , by simple linear relations. A short and elegant derivation of this isomorphism is reproduced in the last section.

Summing up we can say that under the sufficient-ly general assumption of periodic time dependence the following connection exists between electrodynamics and wave mechanics:

$$\boldsymbol{\sigma} \cdot \begin{bmatrix} \text{rot } \mathbf{E} + \frac{\mu}{c} \frac{\partial}{\partial t} \mathbf{H} = \mathbf{0} \\ \text{rot } \mathbf{H} - \frac{\varepsilon}{c} \frac{\partial}{\partial t} \mathbf{E} = \mathbf{0} \\ \text{div } \varepsilon \mathbf{E} = 0 \\ \text{div } \mu \mathbf{H} = 0 \end{bmatrix}_{\text{div } \mathbf{E} = 0} \equiv [(\boldsymbol{\gamma} \cdot \nabla + \boldsymbol{\gamma}^{(4)} \partial_4) \Psi = 0]. \quad (5)$$

In words: Multiplication of source-free electrodynamics by the Pauli-vector yields wave mechanics.

This derivation of wave mechanics appears, from the electrodynamical point of view, as a transformation of the solutions by which the number of unknown component functions is reduced from six to four. In addition, the Pauli-vector generates those symmetries which yield the well-known ease of solution so characteristic of wave mechanics. We may therefore say:

Wave mechanics is a solution-transform of electrodynamics. Here one has to bear in mind that the well-known circulatory structure of the wave functions, manifest in Dirac's hydrogen solution, is not introduced just by the Pauli-vector. The corresponding angular or spin moment exists a priori in source-free electrodynamics and is described its momentum balance [7]

$$\frac{\partial}{\partial t} \mathbf{V} + \text{div } \mathbf{T} = \mathbf{0}, \quad (6)$$

with the momentum density \mathbf{V} and the stress-tensor \mathbf{T} .

2. Epistemological Perspectives

The epistemological perspectives of the Maxwell-Dirac isomorphism are in accord with the tenets of logical positivism. The latter doubts, as one knows, the existence of particles or particle like matter. It

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regards them as illusions, as “things of the imagination” according to Mach [8–11]: “Not each existing scientific theory arises (as) naturally and unaffectedly. When, e.g., chemical, electrical, optical phenomena are explained through atoms, then the auxiliary concept of atoms has not resulted from the principle of continuity, it rather has been invented especially for that purpose. We can nowhere observe atoms, they are things of the imagination, as all substances are. And to atoms even such properties are partially attributed which contradict all those observed so far. May the atomic theories still be suited for describing a number of facts, the natural scientists who took Newton’s rules of philosophizing to heart will accept any such theory only as a provisional aid and will strive for a replacement by a more natural notion.”

An alternative to particle matter is field matter, in case of the Maxwell-Dirac isomorphism an electrodynamic kind of matter: Two photon fields “circle around each other, their centers of gravity forming a Kepler system. In the neighbourhood of the geometric center of this homeopolar system the field strengths reach high intensities and convey the impression of being materialized. Here, the hydrogen atom is a standing disturbance of a special configuration, in essence a disturbance like stationary light.

“To atoms even such properties are partially attributed which contradict all those observed so far.” Here Mach addresses the wellknown contradictions inherent in the particle oriented atomism of his time. His critique fits, however, in a completely analogous way the wave models of Schrödinger and Dirac originating from some time after him, because this wave picture of the twenties was somehow turned back by the conventional Copenhagen interpretation to the Rutherford-Bohr model of the atom. The particles of Rutherford’s model, “things of the imagination, as all substances are”, which seemed to have been overcome in the wave model, recurred by Born’s interpretation.

There is, e.g. the orbiting electron which in violation of electrodynamics – the most reliable of all physical theories – is not allowed to radiate, despite its continually accelerated revolution. Then, however, in jumping between different states, it is obliged to radiate: an absurdity which has been passed on from the old mechanical models to the undulatory ones. These ultimately connect, in an

artificial way, the purely electromagnetic effects of spectral light with the purely mechanical causes of Bohr’s model. The Maxwell-Dirac isomorphism, on the other hand, connects the spectral light with a source-free standing electrodynamic disturbance as its cause, thus avoiding the contradiction.

3. A Short Version of the Maxwell-Dirac Isomorphism

If one wanted to describe the hydrogen gas by means of electrodynamics one should probably start from the firmly established experience that the hydrogen gas may absorb and reemit electromagnetic energy, and that without external intervention there is no indication for the gas to contain electric charges. Thus we consider the hypothesis which visualizes the gas as a charge-free electromagnetic field as the starting point with the least number of assumptions; and so we try to characterize the field by the covariant Maxwell system

$$\begin{aligned} \text{rot } \mathbf{E} + \frac{\mu}{c} \frac{\partial}{\partial t} \mathbf{H} &= \mathbf{0}, \quad \text{div } \varepsilon \mathbf{E} = 0, \\ \text{rot } \mathbf{H} - \frac{\varepsilon}{c} \frac{\partial}{\partial t} \mathbf{E} &= \mathbf{0}, \quad \text{div } \mu \mathbf{H} = 0; \end{aligned} \quad (7)$$

and the condition of being charge-free by means of the requirement

$$\text{div } \mathbf{E} = 0 \text{ besides } \text{div } \mathbf{H} = 0. \quad (8)$$

Because of this we get for (7) the equations

$$\begin{aligned} \text{rot } \mathbf{E} + \frac{\mu}{c} \frac{\partial}{\partial t} \mathbf{H} &= \mathbf{0}, \quad \text{div } \mathbf{E} = 0, \\ \text{rot } \mathbf{H} - \frac{\varepsilon}{c} \frac{\partial}{\partial t} \mathbf{E} &= \mathbf{0}, \quad \text{div } \mathbf{H} = 0 \end{aligned} \quad (9)$$

with

$$\mathbf{E} \perp \text{grad } \varepsilon, \quad \mathbf{H} \perp \text{grad } \mu. \quad (10)$$

Scalar multiplication of the rot equations in (9) by the Pauli-vector, using the algebraic relation

$$(\boldsymbol{\sigma} \cdot \nabla)(\boldsymbol{\sigma} \cdot \mathbf{A}) = \text{div } \mathbf{A} + i \boldsymbol{\sigma} \cdot \text{rot } \mathbf{A} \quad (11)$$

together with the two div equations, transforms that system into

$$\left\{ \begin{aligned} (\boldsymbol{\sigma} \cdot \nabla)(\boldsymbol{\sigma} \cdot \mathbf{H}) - \frac{\varepsilon}{c} \frac{\partial}{\partial t} (i \boldsymbol{\sigma} \cdot \mathbf{E}) &= \mathbf{0}, \\ (\boldsymbol{\sigma} \cdot \nabla)(\boldsymbol{\sigma} \cdot \mathbf{E}) + \frac{\mu}{c} \frac{\partial}{\partial t} (\boldsymbol{\sigma} \cdot \mathbf{H}) &= \mathbf{0}, \\ \mathbf{E} \perp \text{grad } \varepsilon, \quad \mathbf{H} \perp \text{grad } \mu \end{aligned} \right\} \quad (12)$$

In matrix notation this reads

$$\left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot \nabla - \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \frac{1}{c} \frac{\partial}{\partial t} \right] \begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix} = 0, \quad \left. \begin{matrix} E \perp \text{grad } \varepsilon, \quad H \perp \text{grad } \mu. \end{matrix} \right\} \quad (13)$$

Denoting the quantity on which the differential operators act by Ψ , that is

$$\begin{bmatrix} i(\sigma \cdot E) \\ (\sigma \cdot H) \end{bmatrix} = \begin{bmatrix} iE_3 & iH_- \\ iE_+ & -iE_3 \\ H_3 & H_- \\ H_+ & -H_3 \end{bmatrix} = \begin{bmatrix} \Psi'_1 & \Psi''_1 \\ \Psi'_2 & \Psi''_2 \\ \Psi'_3 & \Psi''_3 \\ \Psi'_4 & \Psi''_4 \end{bmatrix} \quad (14)$$

with $X_{\pm} = X_1 \pm iX_2$, and considering the well-known connection

$$\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} = \gamma \quad (15)$$

between the Pauli and the Dirac matrices, we get for (13) the system

$$\left\{ \begin{array}{l} \left[\gamma \cdot \nabla - \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \frac{1}{c} \frac{\partial}{\partial t} \right] \Psi = 0, \\ E \perp \text{grad } \varepsilon, \quad H \perp \text{grad } \mu. \end{array} \right\} \quad (16)$$

Here one has to bear in mind that each of both columns matrix (14) that is

$$\Psi = \begin{pmatrix} iE_3 \\ iE_1 - E_2 \\ H_3 \\ H_1 + iH_2 \end{pmatrix} \quad \text{and} \quad = \begin{pmatrix} iE_1 + E_2 \\ -iE_3 \\ H_1 - iH_2 \\ -H_3 \end{pmatrix} \quad (17)$$

independently represents a system of functions solving (16). From this, a separation of the time dependence according to

$$\Psi = \psi e^{-i\omega t} \quad (18)$$

finally yields the amplitude equation

$$\left\{ \gamma \cdot \nabla + i \frac{\omega}{c} \begin{pmatrix} \varepsilon \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix} \right\} \Psi = 0. \quad (19)$$

Its agreement with the Dirac amplitude equation

$$\gamma \cdot \nabla + \frac{i}{c\hbar} \begin{pmatrix} \hbar\omega - \Phi + m_0 c^2 & 0 \\ 0 & \hbar\omega - \Phi - m_0 c^2 \end{pmatrix} \Psi = 0 \quad (20)$$

is obvious.

The Eqs. (17), as well as (19) or (20), show in addition that the electrodynamical and the wave mechanical field components are connected by simple linear relations, the same holding true for the refraction (ε, μ) in relation to the potential Φ .

This isomorphism can be checked easily and directly because the eight Eq. (9) may be combined into two systems of four equations each, in the following way:

$$\left\{ \begin{array}{l} \pm i(\text{rot } H)_3 \mp i c^{-1} \varepsilon \dot{E}_3 + \text{div } H = 0, \\ \pm i(\text{rot } H)_1 \mp i c^{-1} \varepsilon \dot{E}_1 - (\text{rot } H)_2 + c^{-1} \varepsilon \dot{E}_2 = 0, \\ \pm i \text{div } \bar{E} - (\text{rot } E)_3 - c^{-1} \mu \dot{H}_3 = 0, \\ \pm i(\text{rot } E)_2 \pm i c^{-1} \mu H_2 + (\text{rot } E)_1 + c^{-1} \mu \dot{H}_1 = 0. \end{array} \right\} \quad (21)$$

Inserting here the first or the second wave function of (17) into the first system (upper signs) or the second one (lower signs), respectively, the wave function of (16) ends up immediately, in both cases, and we are back to Dirac again.

With this proof of the theorem that wave mechanics represents a specialized electrodynamics, a first stage in our investigations into the phenomenological meaning of the Schrödinger function has come to an end. The result seems unambiguous and incompatible with the current doctrine which rests on a particle interpretation.

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